all that the successive derivatives of an analytic function cannot be arbitrary; there must always exist an M and an r so that (25) is fulfilled. In order to make the best use of the inequality it is important that r be judiciously chosen, the object being to minimize the function $M(r)r^{-n}$, where $M(r)$ is the maximum of $|f|$ on $|\zeta - a| = r$.

EXERCISES

1. Compute

$$\int_{|z|=1} e^z z^{-n}\,dz, \qquad \int_{|z|=2} z^n(1-z)^m\,dz, \qquad \int_{|z|=\rho} |z-a|^{-4}\,|dz| \;(|a| \neq \rho).$$

2. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

3. If $f(z)$ is analytic and $|f(z)| \leqq M$ for $|z| \leqq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leqq \rho < R$.

4. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leqq 1/(1 - |z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

5. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

***6.** A more general form of Lemma 3 reads as follows:

Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $\alpha \leqq t \leqq \beta$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_\alpha^\beta \varphi(z,t)\,dt$$

is analytic in z and

$$F'(z) = \int_\alpha^\beta \frac{\partial\varphi(z,t)}{\partial z}\,dt. \tag{26}$$

To prove this represent $\varphi(z,t)$ as a Cauchy integral

$$\varphi(z,t) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{\zeta - z}\,d\zeta.$$

Fill in the necessary details to obtain

$$F(z) = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\,dt\right)\frac{d\zeta}{\zeta - z}$$

and use Lemma 3 to prove (26).